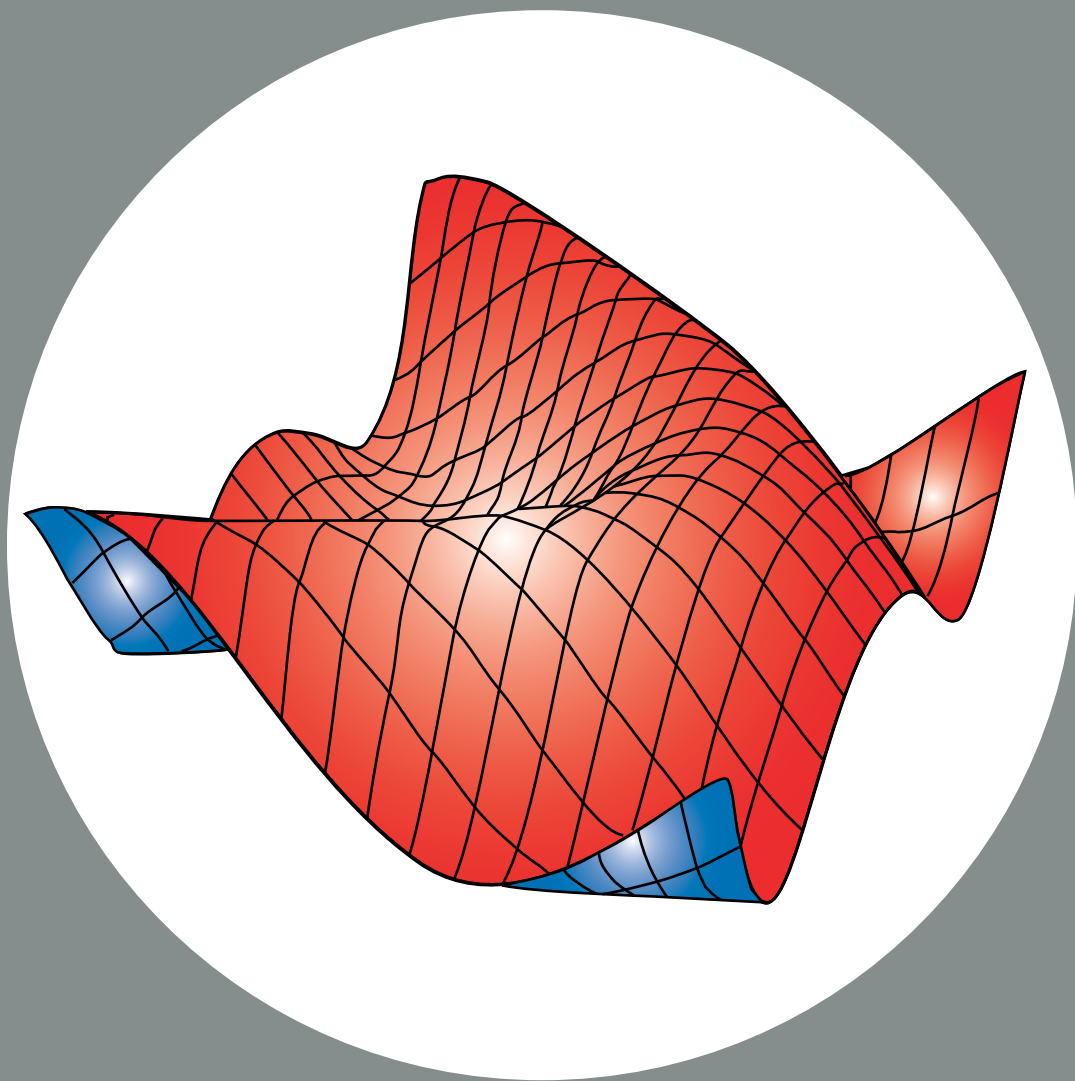


# Trigonometry

Twelfth  
Level of Abstraction



MORENO



# Trigonometry

José Luis Moreno Aranda

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José Luis Moreno Aranda

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# Introduction

This textbook has been created using the Pedagogy of Saint Ignatius of Loyola applied to the teaching-learning of mathematics. This pedagogy contemplates a humanistic philosophy as a philosophical framework of reference, which is to say, it has as its only objective the promotion of the capacities that identify a human as such: imagination, intelligence, creativity, freedom, etcetera. For that reason, the strategic plan of this textbook has been designed solely so that the children that pass through our classrooms may advance in, and may develop fully, their human potential.

The method of Saint Ignatius for internalizing the experience of God<sup>1</sup>, in order to be adapted to the teaching-learning of classroom mathematics as presented here, has been enriched by the epistemology of Bernard Lonergan<sup>2</sup>.

This pedagogical methodology consists of a five-step process that is repeated each time that we introduce a new mathematical concept. This novel methodology is explained in detail in the publication *The Pedagogy of Saint Ignatius of Loyola Applied to the Teaching-learning of Mathematics*<sup>3</sup>.

The pedagogical strategy consists of five steps:

1. *Contextualize the knowledge.* Clearly explain which mathematical concept we are going to study and place it squarely on the upward spiral of knowledge. We must verify that the student has mastered the necessary previous concepts and knows exactly what the next concept studied will be, for the concepts are not isolated entities that suddenly appear, but are like bricks in a wall that form a solid base upon which others will be placed.
2. *Experience a sensory reality through the use of the senses to help the student understand the concept.* As a result of a proper pedagogical strategy, the student, using his or her senses, would touch, see, hear, smell or taste the concept. By mining the data he or she engages in research and engages the imagination, thereby capturing the intelligible unity of that data and subsequently understanding. Using geometry as a guiding theme in the study of mathematics, we have been able to use images and tangible didactic material for the mastery of the principle concepts of mathematics.
3. *Demonstrate or verify the knowledge gained.* When the student has understood, and subsequently has been able to express the concept in his or her own words, only then can we be assured that the student has truly understood. The student, using the mathematical knowledge that he or she has acquired up to this point, demonstrates complete comprehension of the concept.

4. *Apply the knowledge gained and develop the ability to use it in the resolution of different types of problems.* Upon applying the knowledge, the student develops or creates the algorithm that permits him or her to orderly and efficiently solve problems. However, it is not sufficient that the student simply solve the corresponding algorithm; it is also indispensable that he or she develop the ability and accumulate the necessary experience to be able to set out and to solve this type of problem in general.
5. *Evaluate what was learned and how it was learned.* We evaluate the student in two ways. First we must verify that the student has adequately internalized the mathematical concept studied –has shown sufficient ability– and has practiced enough in the setting out and resolving of problems. Subsequently, we must evaluate the way in which we have exposed the student to the concept, which is to say, our own work as teachers.

## Arithmetic and Algebra Books

Mathematiké offers the whole series of arithmetic and algebra textbooks that cover all the formation of a student from first to eleventh grade. All the books have been carefully prepared following the pedagogy of Saint Ignatius of Loyola applied to the teaching-learning of mathematics. All of the arithmetical and algebraic concepts, including basic geometry and its applications are included.

This series of books allow the students to ascend on the spiral of mathematical knowledge in an integral, intelligent and creative way.

## Mathematiké Educational Games

The Mathematiké educational games help the students to use their senses so that they can learn and apply the concepts they study in a simple, engaging and fun way.

The continuous use of games allows the students to develop the skills to apply the concepts.

## Levels of Abstraction

Since the first sensible experiences of a child where touching objects is a privileged way in which he or she is introduced little by little to the marvellous world of natural numbers, to the internalizing of the first mathematical symbols, represent the beginning into the exciting ascent of the development of our imagination or abstraction capacity through mathematics.

The starting point on the spiral of mathematical knowledge is clear and easily identifiable, nonetheless the final point is undefinable since each person will get as far as he or she want to get, and what is most exciting is that during all the time we are alive, we are able to continue ascending. There are no limits or barriers, the ascent is continuous, infinite is unreachable and plenitude is a utopia. When we say that a person keeps on learning and mastering mathematical knowledge is the same as saying that his or her abstraction capacity is increasing, since he or she can imagine more sophisticated mathematical realities.

We can classify the development of the abstraction capacity through mathematical knowledge into four major levels of abstraction:

- Arithmetic.
- Algebra.
- Functions and analytic geometry.
- Infinitesimal mathematics.

## The Objective of this Book

The objective of this book is to cover the first part of the twelfth level of abstraction on the ascending spiral of mathematical knowledge. In this level we use the concepts and their applications of basic and intermediate algebra to enter into the fascinating world of trigonometric functions.

We study the elements of plane geometry to build angles and apply them to triangles.

We study carefully right triangles, Pythagoras theorem, the Cartesian plane, the circle and the trigonometric ratios.

We study the concept of a function using the pedagogic strategy of approaches to the concept.

Using the trigonometric ratios and the concept of a function we build the trigonometric ratios, more trigonometric identities and their applications in trigonometric equations.

## Organization of the Book

The book has been organized by concepts, levels of abstraction and applications of the concepts.

Once the student has finished to go through the concepts and applications of basic and intermediate algebra is ready to go through this book in the order in which the concepts and applications are presented.

## Didactic Material

The appendix contains the didactic material that the student will use throughout the course to understand and prove the concepts of negative numbers, algebraic addition and the use of letters.

Throughout the text there is an explanation on how to use the material.

## Our Web Site

With an aim to maintain direct communication among teachers, students and the research group that has launched this educational initiative, the Mathematiké research project concerning the teaching of mathematics has an internet web site: [www.mathematike.org](http://www.mathematike.org). On this web site we present in much greater detail the pedagogy of Saint Ignatius of Loyola, the complete and up to date list of educational games and the textbooks. We are committed to keeping our educational project up to date, which is why communication with all of you, the teachers, is of vital importance.

1. *Obras de San Ignacio de Loyola*. Biblioteca de autores cristianos, Madrid. 1997.
2. Lonergan, Bernard, *Collected works of Bernard Lonergan*, Insight, vol 3, University of Toronto Press, 1997.
3. Moreno Aranda, José Luis, *Pedagogía de San Ignacio aplicada a la enseñanza de las matemáticas*, Grupo Mathematiké, 2003.